Chapter 3

Random Process

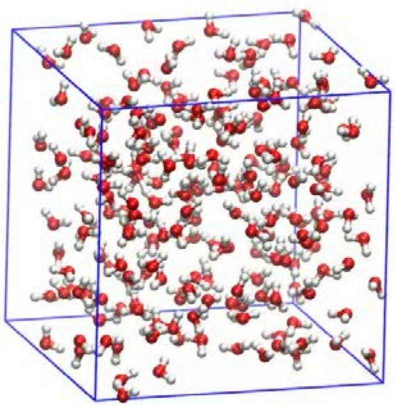
&

Partial Differential Equations

Deterministic model (repeatable results)

Consider N particles with coordinates $\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N$

- the interactions between particles modeled by potential $V(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N)$
- the dynamics of the system represented by ordinary differential equations



$$m_i \frac{d^2\mathbf{r}_i}{dt^2} = \mathbf{f}_i, \qquad i = 1, 2, 3 \cdots N$$

$$\mathbf{f}_i = -\nabla_{\mathbf{r}_i} V(\mathbf{r}_1, \mathbf{r}_2, \cdots \mathbf{r}_N) = \left(-\frac{\partial V}{\partial x_i}, -\frac{\partial V}{\partial y_i}, -\frac{\partial V}{\partial z_i} \right)$$

These coupled equations can be solved using *numerical method*

Verlet algorithm

Taylor expansion

$$\mathbf{r}(t+\Delta t)=\mathbf{r}(t)+\mathbf{v}(t)\Delta t+(1/2)\mathbf{a}(t)\Delta t^2+(1/6)\mathbf{b}(t)\Delta t^3+\underline{O(\Delta t^4)}$$

$$\mathbf{r}(t-\Delta t)=\mathbf{r}(t)-\mathbf{v}(t)\Delta t+(1/2)\mathbf{a}(t)\Delta t^2-(1/6)\mathbf{b}(t)\Delta t^3+\underline{O(\Delta t^4)}$$

$$\Downarrow$$

相加 $\mathbf{r}(t+\Delta t)=2\mathbf{r}(t)-\mathbf{r}(t-\Delta t)+\mathbf{a}(t)\Delta t^2+\underline{O(\Delta t^4)}$

相减 $\mathbf{v}(t)=\dfrac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t-\Delta t)}{2\Delta t}+\underline{O\left(\Delta t^2\right)}$

Verlet integrator is an order more accurate than integration by simple Taylor expansion alone, with the same term Δt^2

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3.0 Introduction

velocity Verlet algorithm

Verlet algorithm is not self-starting, we will use velocity Verlet algorithm in molecular dynamics simulations.

$$\mathbf{r}(t+\Delta t) = \mathbf{r}(t) + \mathbf{v}(t)\Delta t + (1/2)\mathbf{a}(t)\Delta t^2 + O\left(\Delta t^3\right)$$

$$\mathbf{v}\left(t+\Delta t\right) = \mathbf{v}\left(t\right) + \frac{\mathbf{a}\left(t\right)+\mathbf{a}\left(t+\Delta t\right)}{2}\Delta t + O\left(\Delta t^2\right)$$

Untitled.cdf

Probabilistic model (unrepeatable results)

Consider the same system of N particles as before

- when there is uncetainty, say $random\ force\ \mathbf{R}_i$, which has a Gaussian probability distribution with correlation function

$$\left\langle \mathbf{R}_i(t)\cdot\mathbf{R}_i(t')\right\rangle = 6\gamma k_B T\delta(t-t')$$

γ: friction coefficient

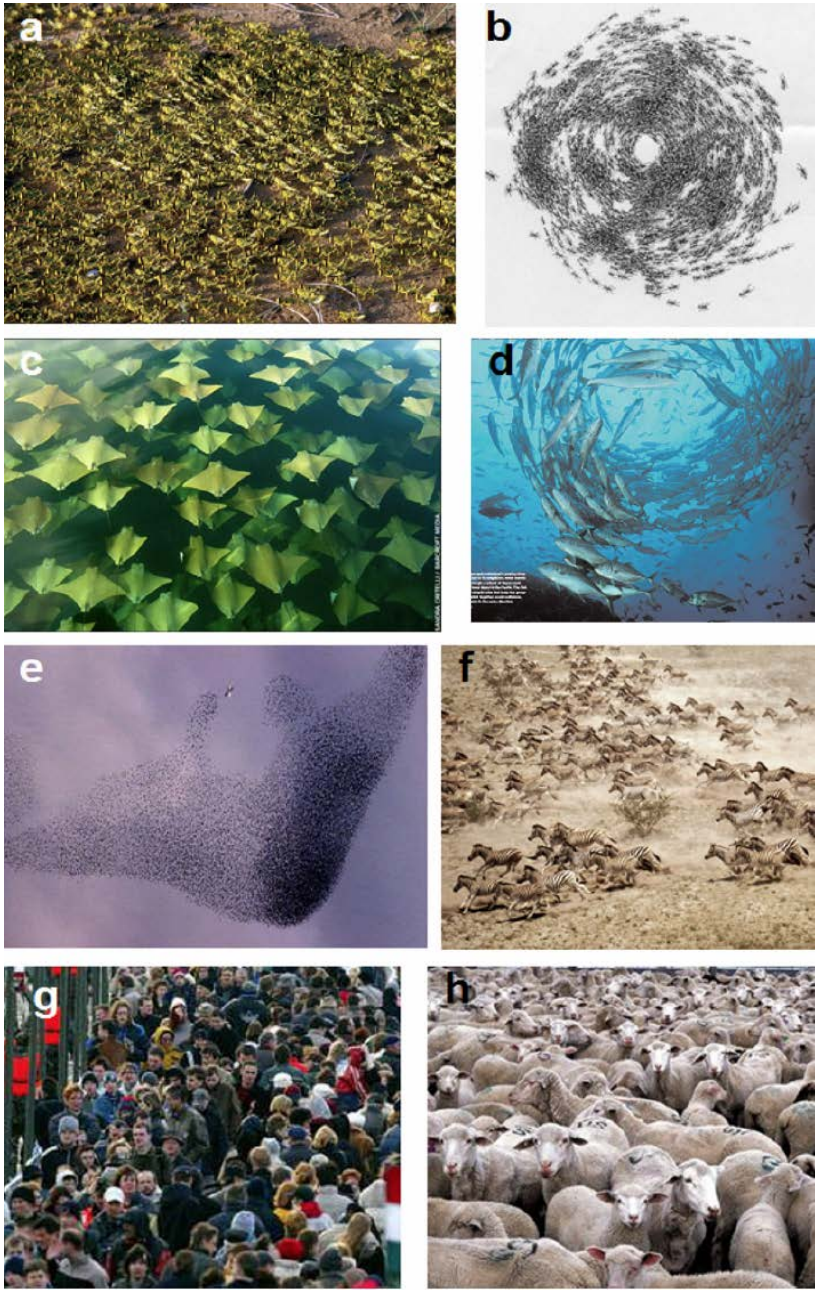
Langevin Equation:

$$m_i\frac{d^2\mathbf{r}_i}{dt^2}+\gamma\frac{d\mathbf{r}_i}{dt}=\mathbf{f}_i+\mathbf{R}_i(t),\ \ i=1,2,3\cdots N$$

$$\mathbf{f}_i=-\nabla_{\mathbf{r}_i}V(\mathbf{r}_1,\mathbf{r}_2,\cdots\mathbf{r}_N)=\left(-\frac{\partial V}{\partial x_i},-\frac{\partial V}{\partial y_i},-\frac{\partial V}{\partial z_i}\right)$$

Each particle's motion can be described by certain probabilities, derived from *Fokker- Planck Equation.*

3.0 Introduction



(a) Wingless Locusts marching in the field.

(b) A rotating colony of army ants.

(c) A three-dimensional array of golden rays.

(d) Fish are known to produce such vortices.

(e) Before roosting, thousands of starlings producing a fascinating aerial display.

(f) A herd of zebra.

(g) People spontaneously ordered into traffic lanes as they cross a pedestrian bridge in large numbers.

(h) Although sheep are known to move very coherently, just as the corresponding theory predicts, when simply hanging around (no motion), well developed orientational patterns cannot emerge.

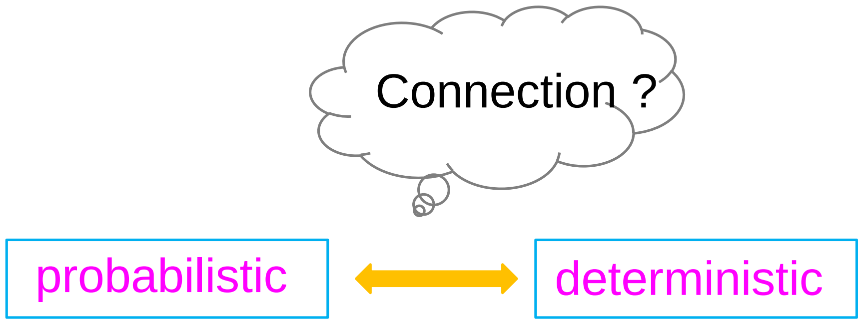
Tamás Vicsek, **Collective motion**,
Physics Reports, Vol 517, 2012, Pages 71-140

3.0 Introduction

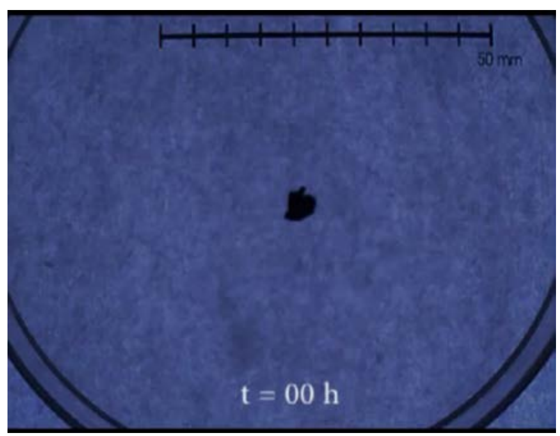
3.0 Introduction

our primary goal:

- to investigate the connection between probabilistic and deterministic models of the same phenomenon.



Micro view:
single random process



Macro view: (ensemble average)
definite distribution function

described by partial
differential equation

- Looks paradoxical that a random process can be characterized by a definite equation
- But we know it is true from a lot of daily experience, such as coin tossing.

coin tossing

~50% ~50%

many parameters unknown

- the initial orientation, velocity, and spin;
- the properties of the table surface;
- Various atomic defects, dislocations, grains, voids…

■ For single toss: no idea whether head or tail turns up.

■ After a large number of tosses, proportion of heads or tails is ~0.5.

■ With this example, it is not so surprising that there is a determinable distribution of probabilities which characterizes a random process.

3.0 Introduction

Probabilistic model originates from

- Incompleteness of information

  e.g. coin tossing

- Parameters sensitivity --- tiny perturbation in input induces huge variation in output.

  e.g. In kinetics of gases, a slight change in the initial conditions would result in a tremendous change after many collisions.

By an averaging of the solutions with varying initial conditions random processes can be modeled successfully.

3.0 Introduction

in the following sections,

- <u>Section 3.1</u>
  1-D Brownian motion. An explicit expression of the probability $w(m, N)$.

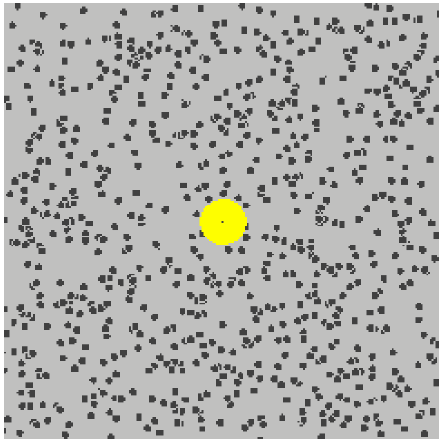
- <u>Section 3.2</u>
  Simplified expression of $w(m, N)$. Asymptotic Series, Laplace's Method

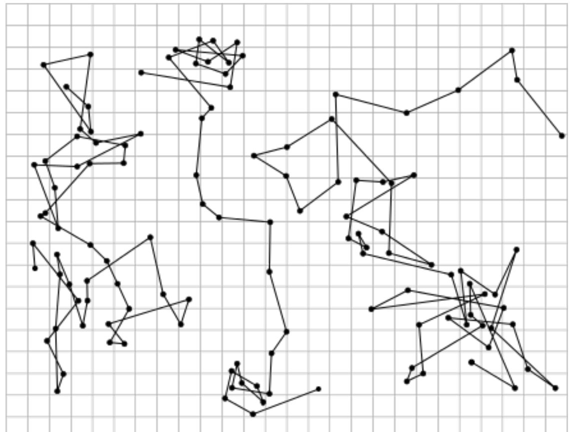
- <u>Section 3.3</u>
  a difference equation for $w(m, N)$ leads to a partial differential equation.

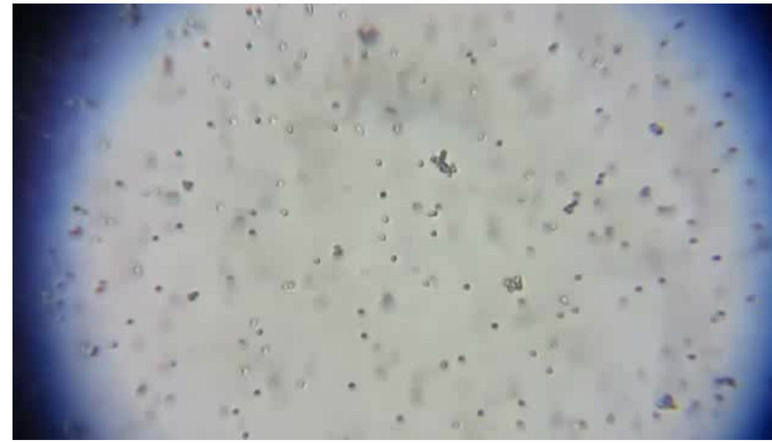
- <u>Section 3.4</u>
  The connection between probability and differential equations.

3.1 Random Walk in One Dimension; Langevin's Equation

In Brownian motion, small particles move about in liquid or gas.

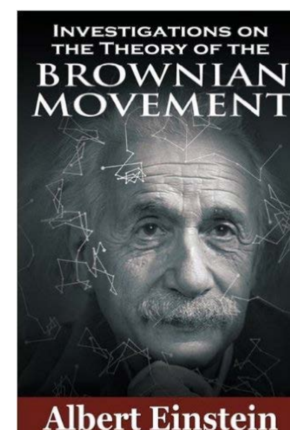






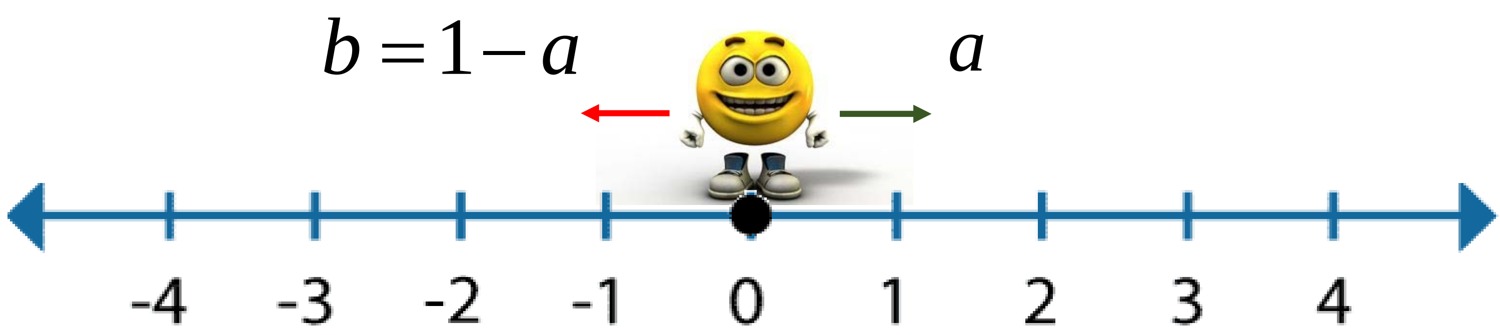
- no possibility and no interest of computing the trajectory of each molecule.
- one wishes to have an average understanding of the phenomenon.

- The Roman Lucretius's (卢克莱修) described Brownian motion of dust in his scientific poem "On the Nature of Things" (60 BC)
- Botanist Robert Brown in 1827 studied pollen grains suspended in water.
- Albert Einstein in 1905 solved this problem.



3.1.1 An one dimension random walk model

The minimum model of random walk: 1-D lattice model



Particle moves according to the following rules:

- Move in steps of a fixed length *dx* in a fixed time interval *dt*.
- The probability to the right *p* and to the left *q=1-p*.

Goal: to obtain the probability $w(m, N)$

- m steps to the right of the origin
- N the total steps

$$\text{Probability} = \frac{\text{number of observed events}}{\text{total number of events}}$$

3.1.1 An one dimension random walk model

random walk model can be found in various situations

✓ a drunk staggering down a street

✓ a gambling game in which a coin is tossed

✓ Polymer Physics: Freely jointed chain model

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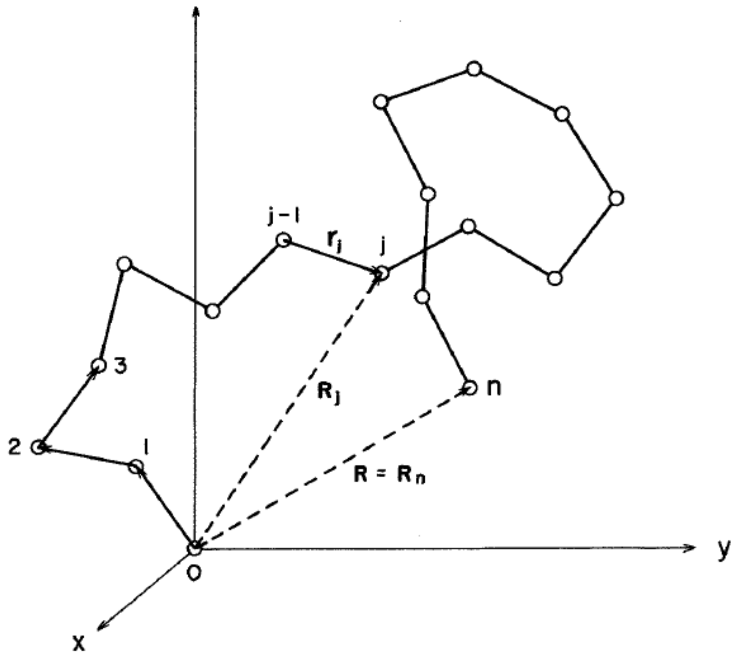
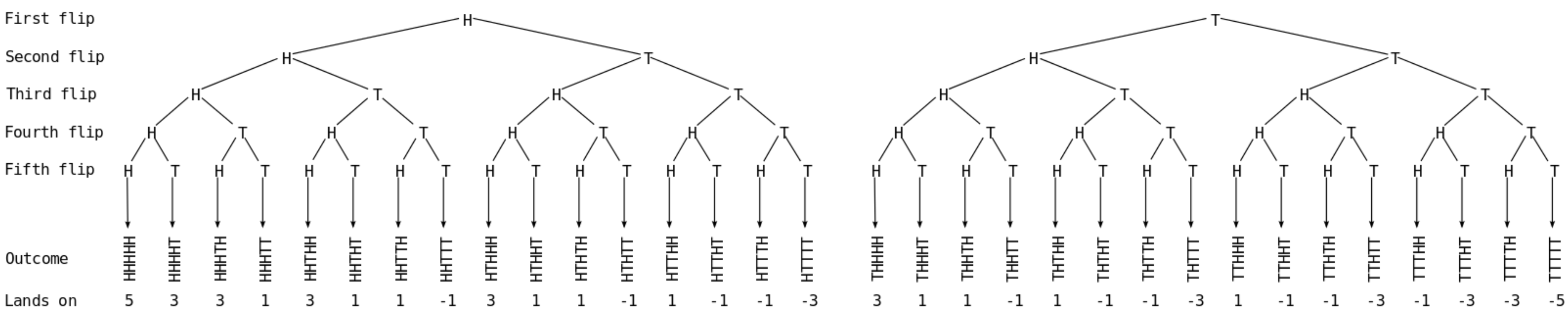
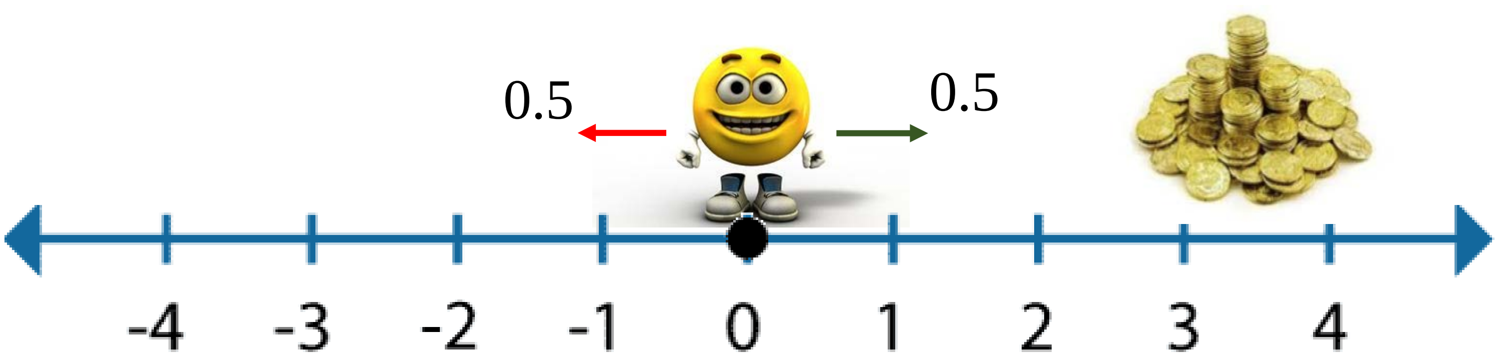


3.1.1 An one dimension random walk model

Fair coin tossing

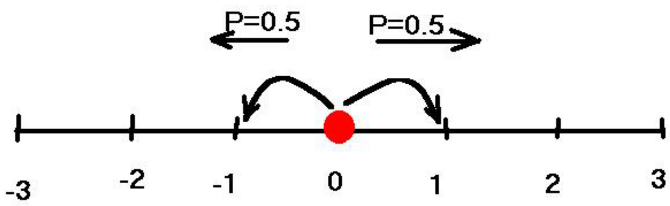
Head ~50% Tail ~50%

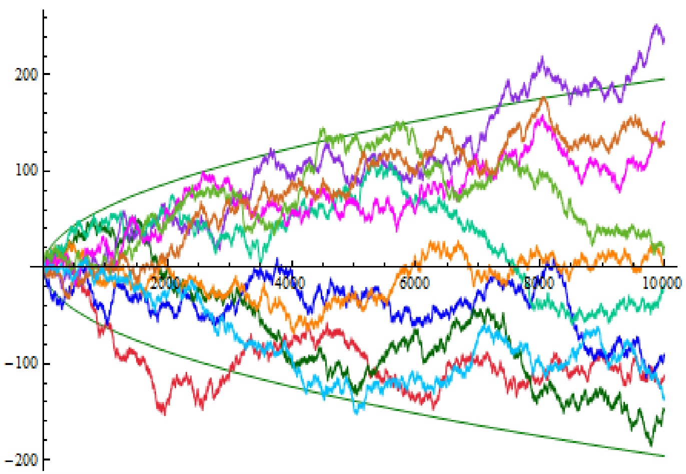


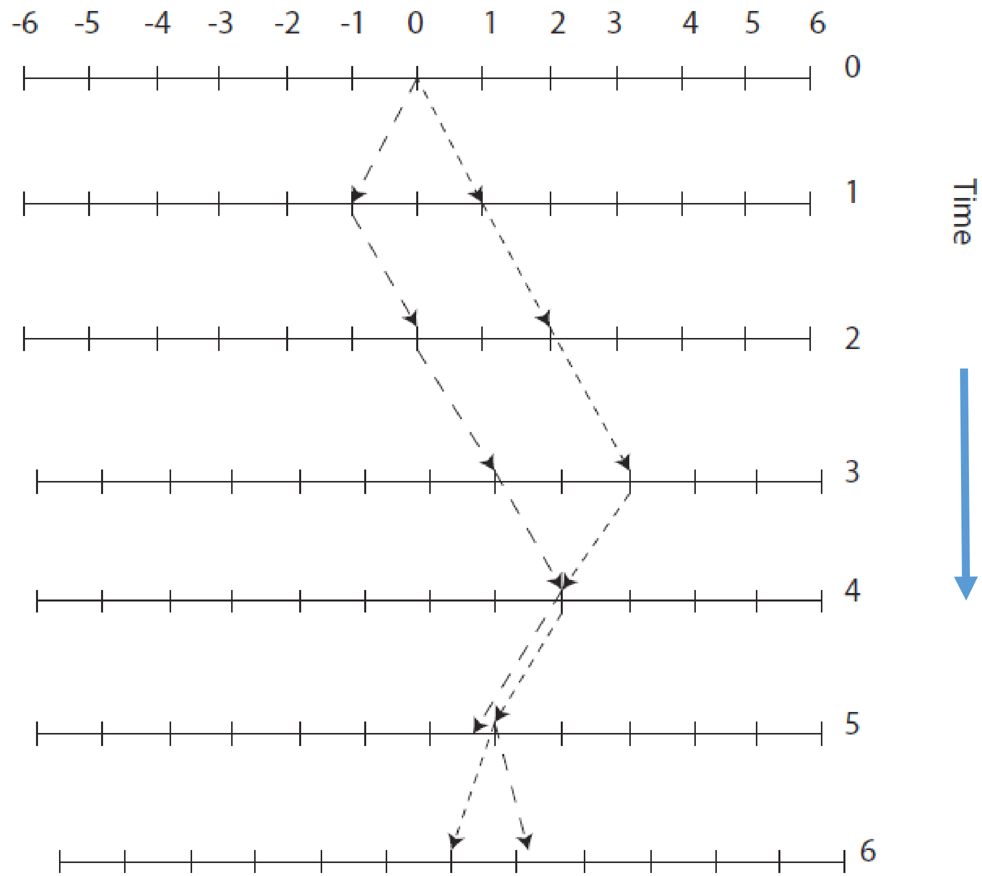


3.1.1 An one dimension random walk model

Random walk



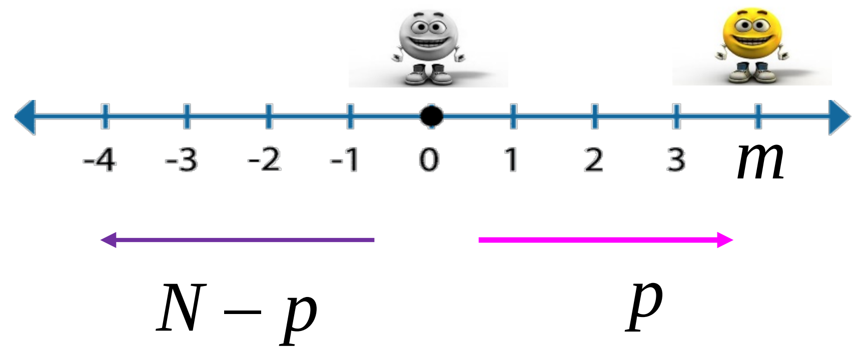




3.1.2 Explicit solution

To find $w(m, N)$, the probability that a particle at a point $m \in [-N, N]$ steps to the right of its origin after total N steps.

Suppose that the particle
- p steps to the right, $p>0$
- $N-p$ steps to the left



Displacment m

$$m = p - (N-p) = 2p - N$$

$$p = (N + m)/2$$

e.g.	N=12	N-p=5	p=7	m=2

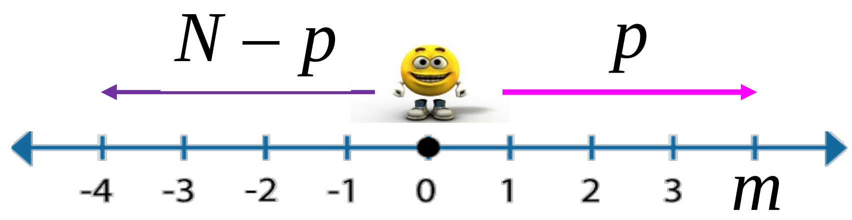
N is even $\rightarrow$ m is even. N is odd $\rightarrow$ m is odd

For example,

if N=3, the possible values of m = -3, -1, 1, 3.

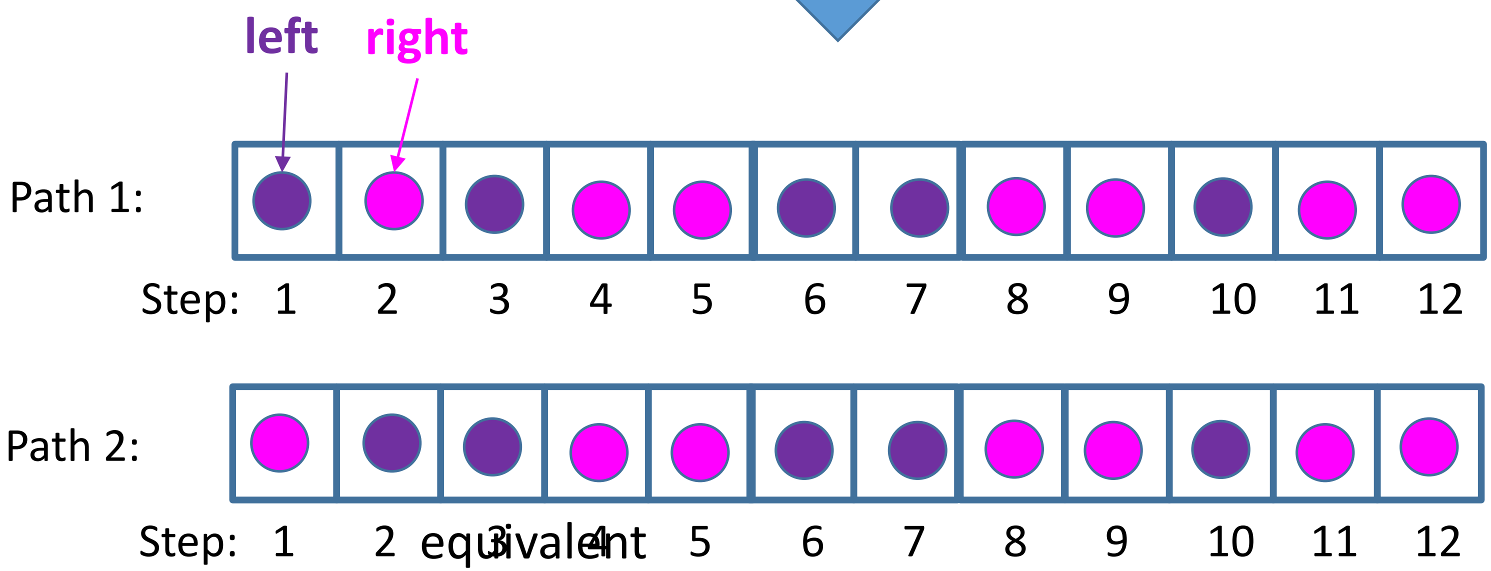
if N=4, the possible values of m = -4, -2, 0, 2, 4.

3.1.2 Explicit solution



To find out the number of paths with ***p* steps to the right** and ***N - p* to the left.**

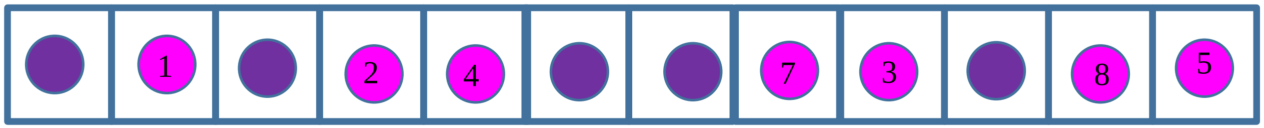
equivalent



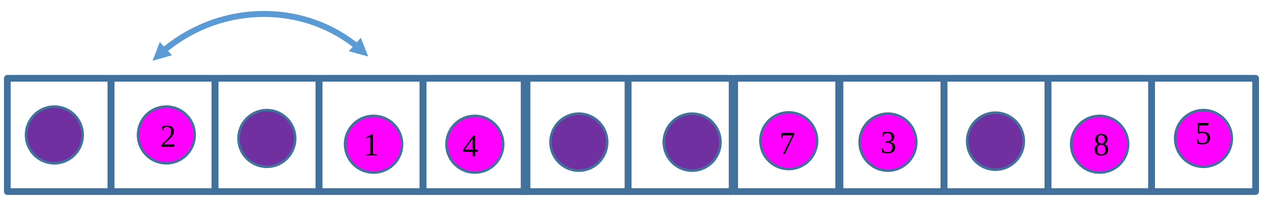
The number of choices with ***p*** indistinguishable **pink ball in *N* boxes**

3.1.2 Explicit solution

Consider *p* distinguishable balls, which can be placed in *N* boxes in the following number of ways:



$$N(N-1)(N-2)\cdots(N-p+1)=\frac{N!}{(N-p)!}$$



Interchanging distinguishable balls does not change the pattern.
There are $p!$ permutations of *p* balls.

排列

3.1.2 Explicit solution

number of ways *p* distinguishable balls can be placed in *N* boxes	=	number of full box empty box patterns	$\times$	number of full permutations of distinguishable balls within a pattern

$$\frac{N!}{(N-p)!} = C_p^N \times p!$$

binomial coefficient $$C_p^N = \frac{N!}{p!(N-p)!}$$

$$(x+y)^N = \sum_{p=0}^{N} C_p^N x^{N-p} y^p$$

- The total number of possible path is 2^N
- the probability that a particle at a point *m* steps to the right of its origin after total *N* steps.

$$\boxed{w(m, N) = \frac{C_p^N}{2^N}}$$

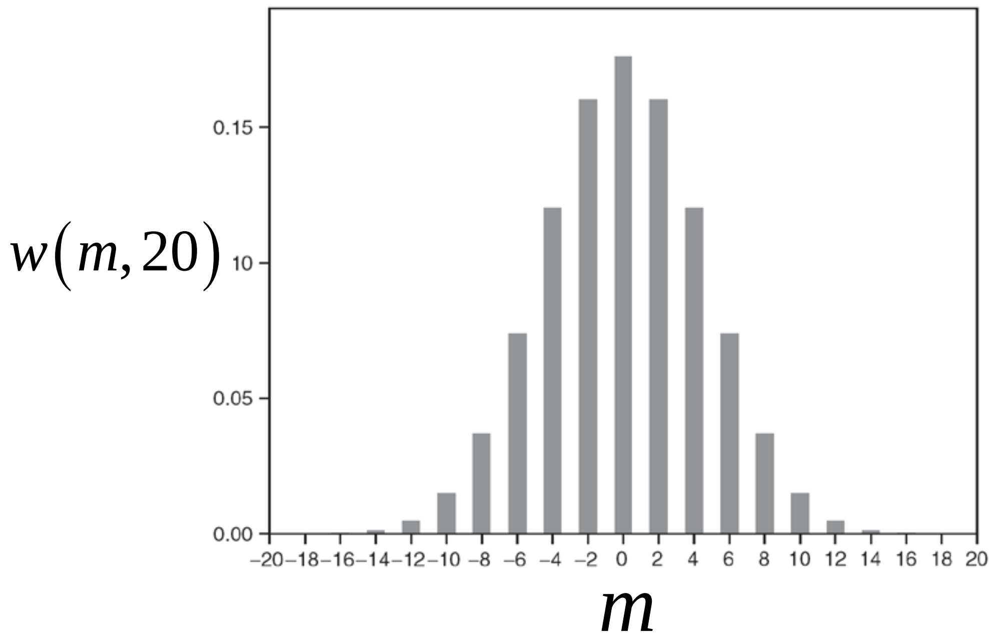
where $p = (N + m)/2$

The sum of all probabilities is unity

$$\sum_{m=-N}^{N} w(m,N) = \sum_{p=0}^{N} C_p^N \left(\frac{1}{2}\right)^N$$

$$= \sum_{p=0}^{N} C_p^N \left(\frac{1}{2}\right)^{N-p} \left(\frac{1}{2}\right)^p$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right)^N = 1$$

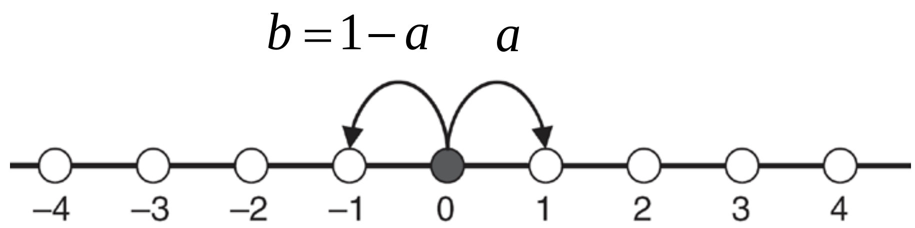


3.1.2 Explicit solution

Characteristic functions

- the **characteristic function** of any real-valued random variable defines its probability distribution.

Let $\lambda(\theta) = ae^{i\theta} + be^{-i\theta}$



$$\lambda^2(\theta) = \left(ae^{i\theta} + be^{-i\theta}\right)^2 = a^2 e^{i2\theta} + 2abe^{i0} + b^2 e^{-i2\theta}$$

	m=2	m=0	m=-2
$P_N(m)$	$P_2(2)$	$P_2(0)$	$P_2(-2)$
	$= a \cdot a$	$= 2a \cdot b$	$= b \cdot b$

these coefficients are the probability

characteristic function $\lambda(\theta) = ae^{i\theta} + be^{-i\theta} = \langle e^{i\theta x} \rangle$ $\quad x = \pm 1$

mean

3.1.2 Explicit solution

To extract the coefficient analytically , for example as m=2

$$\lambda^2(\theta) = \left(ae^{i\theta} + be^{-i\theta}\right)^2 = a^2 e^{i2\theta} + 2abe^{i0} + b^2 e^{-i2\theta}$$

$$P_2(2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda^2(\theta)\, e^{-i2\theta} d\theta$$

Fourier transform

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(a^2 e^{i2\theta} + 2abe^{i0} + b^2 e^{-i2\theta}\right) e^{-i2\theta} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(a^2 e^{i0\theta} + 2abe^{-i2\theta} + b^2 e^{-i4\theta}\right) d\theta$$

$$= \frac{1}{2\pi}\left(a^2 \int_{-\pi}^{\pi} d\theta + 2ab\int_{-\pi}^{\pi} e^{-i2\theta} d\theta + b^2 \int_{-\pi}^{\pi} e^{-i4\theta} d\theta\right)$$

$$= a^2$$

3.1.2 Explicit solution

Generally, we extract the coefficient via Fourier transform

$$P_N(m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda^N(\theta) e^{-i\theta m} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(ae^{i\theta} + be^{-i\theta}\right)^N e^{-i\theta m} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=0}^{N} C_k^N a^{N-k} e^{i\theta(N-k)} b^k e^{-i\theta k} e^{-i\theta m} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=0}^{N} C_{N-k}^N a^{N-k} e^{i\theta(N-k)} b^k e^{-i\theta k} e^{-i\theta m} d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{p=0}^{N} C_p^N a^p e^{i\theta p} b^{N-p} e^{-i\theta(N-p)} e^{-i\theta m} d\theta$$

$$= \frac{1}{2\pi}\sum_{p=0}^{N} C_p^N a^p b^{N-p} \int_{-\pi}^{\pi} e^{i\theta(2p-N-m)} d\theta$$

$$C_p^N = C_{N-p}^N$$

$$p = N - k$$

3.1.2 Explicit solution

we notice

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta(2p-N-m)}d\theta = \begin{cases} 0, & if \;\; p \neq (N+m)/2 \\ 1, & if \;\; p = (N+m)/2 \end{cases}$$

Thus, we have

$$\boxed{P_N(m) = C_p^N a^p b^{N-p}, \qquad p = (N+m)/2}$$

Specifically,

$$P_N(m) = w(m,N) = \frac{C_p^N}{2^N}$$

when $a = b = 1/2$

General Characteristic function

- probability density function (PDF) $p(x)$.
- displacement x is continuous.
- the characteristic function is given by

$$\lambda(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} p(x) e^{ikx} dx$$

$$\lambda(\theta) = ae^{i\theta} + be^{-i\theta} = \langle e^{i\theta x} \rangle$$

Fourier transform of $p(x)$.

- One important property of the characteristic function

$$\lambda(0) = \int_{-\infty}^{\infty} p(x) dx = 1$$

3.1.2 Explicit solution

General random walk model

- Consider a continuous 1-D random walk process of n steps
- we have recursion relation:

convolution

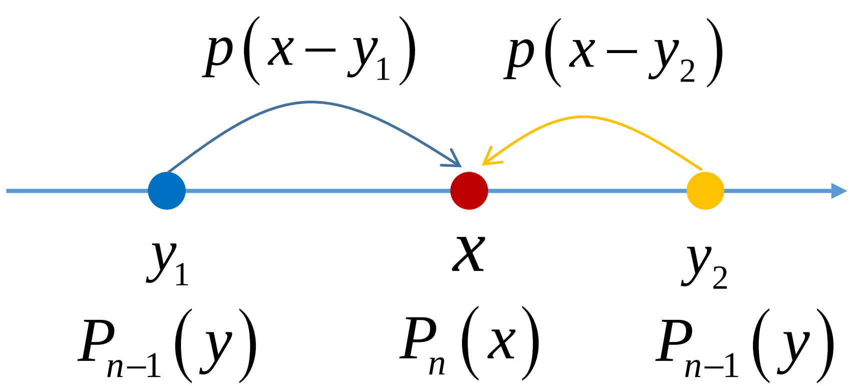
$$f(x)*g(x)=\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy$$

$$P_n(x)=\int_{-\infty}^{\infty} P_{n-1}(y)\,p(x-y)\,dy$$

$$=P_{n-1}(x)*p(x)$$

This means that the probability $P_n(x)$ of a particle at x after *n* steps is

- $P_{n-1}(y)$ the probability of arriving at y in n −1 steps
- $p(x-y)$ the probability of displacements x-y in one step.



$$P_n(x)=\sum_i P_{n-1}(y_i)\,p(x-y_i)$$

$$\rightarrow P_n(x)=\int_{-\infty}^{\infty} P_{n-1}(y)\,p(x-y)\,dy$$

3.1.2 Explicit solution

Let us define

$$P_n(k)=\int_{-\infty}^{\infty}P_n(x)e^{ikx}dx$$

$$\Downarrow$$

$$P_n(k)=P_{n-1}(k)\lambda(k)$$

$$\Downarrow$$

$$P_n(k)=P_{n-1}(k)\lambda(k)=P_{n-2}(k)\lambda^2(k)=\cdots=\lambda^n(k)$$

$$\Downarrow$$

$$\boxed{P_n(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}P_n(k)e^{-ikx}dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda^n(k)e^{-ikx}dx}$$

convolution theorem

$$f(x)*g(x)=\int_{-\infty}^{\infty}f(y)g(x-y)dy$$

$$F\left[f(x)*g(x)\right]=\frac{1}{2\pi}f(k)\cdot g(k)$$

$$P_n(x)=P_{n-1}(x)*p(x)$$

$$\lambda(k)=\int_{-\infty}^{\infty}p(x)e^{ikx}dx$$

3.1.3 Mean, Variance, and the Generating function

The expected value of function f is defined by

$$\langle f \rangle = \sum_{m=-N}^{N} f(m) w(m, N)$$

$$\langle p \rangle = \sum_{m=-N}^{N} p(m) w(m, N) = \sum_{p=0}^{N} p C_p^N \left(\frac{1}{2}\right)^N$$

n-th moment $\langle m^n \rangle = \sum_{m=-N}^{N} m^n w(m, N) = \sum_{m=-N}^{N} m^n C_p^N \left(\frac{1}{2}\right)^N$ where $p = (N + m)/2$

$\langle m \rangle$ mean displacement

$\langle m^2 \rangle$ mean square displacement or variance

3.1.3 Mean, Variance, and the Generating function

In order to evaluate the various moments, we introduce the generating function

$$G(u)=\sum_{p=0}^{N}u^{p}w(m,N)$$

or $$G(u)=\sum_{p=0}^{N}u^{p}C_{p}^{N}\left(\frac{1}{2}\right)^{N}=\sum_{p=0}^{N}C_{p}^{N}u^{p}\left(\frac{1}{2}\right)^{N-p}\left(\frac{1}{2}\right)^{p}=(1+u)^{N}\left(\frac{1}{2}\right)^{N}$$

Example: to calculate $\langle m\rangle$

$$G'(u)=\sum_{p=0}^{N}pu^{p-1}w(m,N)\Rightarrow G'(1)=\sum_{p=0}^{N}pw(m,N)=\langle p\rangle$$

$$G(u)=(1+u)^{N}\left(\frac{1}{2}\right)^{N}\Rightarrow G'(1)=N/2$$

$$\Rightarrow \quad \langle p\rangle=N/2$$

since $p=(N+m)/2$

$$\langle p\rangle=\frac{1}{2}\sum_{p=0}^{N}(N+m)w(m,N)$$

$$=\frac{1}{2}\sum_{p=0}^{N}Nw(m,N)+\frac{1}{2}\sum_{p=0}^{N}mw(m,N)=\frac{N}{2}+\frac{\langle m\rangle}{2}$$

$$\Rightarrow \quad \langle m\rangle=0$$

3.1.3 Mean, Variance, and the Generating function

Example: $\langle m^2 \rangle^{1/2} = ?$

$$G'(u) = \sum_{p=0}^{N} p u^{p-1} w(m,N) \qquad G''(u) = \sum_{p=0}^{N} p(p-1) u^{p-2} w(m,N)$$

$$G''(1) = \sum_{p=0}^{N} p(p-1) w(m,N) = \langle p(p-1) \rangle = \langle p^2 \rangle - \langle p \rangle$$

$$G(u) = (1+u)^N \left(\frac{1}{2}\right)^N \Rightarrow G''(u) = N(N-1)(1+u)^{N-2} \left(\frac{1}{2}\right)^N$$

$$\left.\begin{aligned} G''(1) &= \frac{N(N-1)}{4} \\ G''(1) &= \langle p^2 \rangle - \langle p \rangle \end{aligned}\right\} \quad \langle p^2 \rangle = \langle p \rangle + \frac{N(N-1)}{4} = \frac{N^2}{4} + \frac{N}{4}$$

$m = 2p - N$

$$\langle m^2 \rangle = \langle (2p-N)^2 \rangle = 4\langle p^2 \rangle + N^2 - 4N\langle p \rangle = N^2 + N + N^2 - 4N\frac{N}{2} = N$$

$$\boxed{\langle m^2 \rangle^{1/2} = N^{1/2}}$$

3.1.3 Mean, Variance, and the Generating function

Generally, the nth- moment $\langle x^n \rangle = \int_{-\infty}^{\infty} p(x) x^n dx$

Its characteristic function

$$\lambda(k) = \langle e^{ikx} \rangle = \int_{-\infty}^{\infty} p(x) e^{ikx} dx = \int_{-\infty}^{\infty} dx\; p(x) \left(1 + ikx - \frac{k^2 x^2}{2!} + i \frac{k^3 x^3}{3!} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \frac{i^n k^n}{n!} \langle x^n \rangle$$

We obtain n-th moments using characteristic function

$$\langle x^n \rangle = (-i)^n \left. \frac{d^n \lambda(k)}{dk^n} \right|_{k=0}$$

3.1.4 To determine Boltzmann's constant from Brownian Motion

Theory by Einstein, experiment by Perrin

Einstein

- Assume that the macroscopic resistance on the particle is proportional to the velocity - by classical hydrodynamics
- showed diffusion obey the statistical law

$$\langle x^2 \rangle = \frac{1}{3}\left[\langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle\right] = \frac{1}{3}\langle r^2 \rangle = 2Dt$$ Verified by Perrin

the diffusion coefficient D is given by

$$D = kT / f$$

T : absolute temperature; K : Boltzmann's constant

f : the coefficient of resistance

$$f = 6\pi\mu a \qquad \text{(Stokes' law)}$$

μ : viscosity coefficient; a : particle size

3.1.4 To determine Boltzmann's constant from Brownian Motion

The modern theory of the Brownian motion

Langevin's equation $$m\frac{d\mathbf{v}}{dt} = -f\mathbf{v} + \mathbf{F}(t)$$

where ***v*** the velocity of the particle and *m* mass. The random force follows Fluctuation-dissipation relation

$$\left\langle \mathbf{F}_i(t) \cdot \mathbf{F}_i(t') \right\rangle = 6 f k_B T \delta(t - t')$$

3.1.4 To determine Boltzmann's constant from Brownian Motion

To solve $$m\frac{d\mathbf{v}}{dt} = -f\mathbf{v} + \mathbf{F}(t)$$

multiply with **x**, and take the ensemble average

$$m\left\langle \mathbf{x}\cdot\frac{d\mathbf{v}}{dt}\right\rangle = -f\langle \mathbf{x}\cdot\mathbf{v}\rangle + \langle \mathbf{x}\cdot\mathbf{F}(t)\rangle$$

$$\Downarrow$$

$$m\left(\frac{d\langle \mathbf{x}\cdot\mathbf{v}\rangle}{dt} - \langle v^2\rangle\right) = -f\langle \mathbf{x}\cdot\mathbf{v}\rangle + \langle \mathbf{x}\cdot\mathbf{F}(t)\rangle$$

$$\langle \mathbf{x}\cdot\mathbf{F}(t)\rangle = 0$$

Not correlated

$$\Downarrow$$

$$\frac{d\langle \mathbf{x}\cdot\mathbf{v}\rangle}{dt} + \frac{f}{m}\langle \mathbf{x}\cdot\mathbf{v}\rangle - \langle v^2\rangle = 0$$

$$\Downarrow$$

$$\langle \mathbf{x}\cdot\mathbf{v}\rangle = ce^{-\frac{f}{m}t} + \frac{m}{f}\langle v^2\rangle \rightarrow \frac{m}{f}\langle v^2\rangle \text{ stationary solution}$$

3.1.4 To determine Boltzmann's constant from Brownian Motion

$$\left\langle x^2 \right\rangle = \frac{1}{3}\left\langle r^2 \right\rangle = 2Dt$$

$$\left\langle \mathbf{x}\cdot\mathbf{v} \right\rangle = \frac{1}{2}\frac{d\left\langle \mathbf{x}\cdot\mathbf{x} \right\rangle}{dt} = \frac{1}{2}\frac{d\left\langle r^2 \right\rangle}{dt} = \frac{1}{2}\frac{d\left(6Dt\right)}{dt} = 3D$$

$$\left\langle \mathbf{x}\cdot\mathbf{v} \right\rangle = \frac{m}{f}\left\langle v^2 \right\rangle$$

$$\frac{1}{2}m\left\langle v^2 \right\rangle = \frac{1}{2}f\left\langle \mathbf{x}\cdot\mathbf{v} \right\rangle = \frac{3}{2}fD$$

$$\frac{1}{2}m\left\langle v^2 \right\rangle = \frac{3}{2}kT$$

energy equipartition principle

$\Rightarrow$

Boltzmann constant

$$k = D\frac{f}{T} = D\frac{6\pi\mu a}{T}$$

$$f = 6\pi\mu a$$

作业： Ex. 4　Page 90

修正：Eq.(24) 应该为

$$J_p(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p}$$